NATIONAL BUREAU OF STANDARDS REPORT

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OPTIMAL NETWORKS JOINING n POINTS IN THE PLANE

by

W.A. Horn

Technical Report

to

Northeast Corridor Transportation Project

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p. 20 : Last two sentences of the proof of theorem 12 should be replaced by (new paragraph):

Now by theorem 11, if N is any network with more than $\frac{1}{4}n(n-1)^2(n-2)$ auxiliary nodes, we may find a subnetwork of N connecting all initial nodes, containing a minimal path between each pair of initial nodes, and having $\leq \frac{1}{4}n(n-1)^2(n-2)$ auxiliary nodes of order ≥ 3. Clearly this subnetwork does not contain all arcs of the original network and so, since $\lambda > 0$, γ is smaller for the subnetwork. Thus we need consider only a finite number of values for n_a, namely the integers from 0 to $\frac{1}{4}n(n-1)^2(n-2)$. Hence there exists an optimal network.

p. 21 : Delete last two lines of statement of lemma 13, and replace by:

with costs of travel the same as in the original, except that the cost between x and any other initial node p_j is λ_{ij}.

p.22, ℓ.4 : Change "demands" to "travel costs".

p.25, ℓ.1 : Comma after "nodes".

p.26, ℓ. 6^-: Change A" to A'.

p.27, fig.5: Letter "D" is missing from figure. Insert at same location as in figure 4.

p. 32: Delete first two sentences in proof of lemma 19. Replace by:

From corollary 8 we have that the vectors directed outward from the Y in the direction of its incident arcs and having lengths equal to the weights carried by the respective arcs sum to the zero vector. Furthermore if the vector addition is represented graphically, a triangle will be formed such that the angle between any two adjacent sides is the supplement of the angle made by the arcs of the Y corresponding to those two sides.

p. 36: Add the following sentence to the end of the paragraph preceding corollary 21:

Its statement here uses the normalization $\lambda + \Sigma \lambda_{ij} = 1$, mentioned previously.

p. 38, ℓ.2: $\frac{1}{2}\ 1-\lambda)^2$ should be $\frac{1}{2}(1-\lambda)^2$.

p. 42, ℓ.13: Delete "according to the author." End sentence with a period.

ABSTRACT

This report develops a number of results on the problem of connecting n points in the plane, with given travel demands between each pair, by a minimum-cost network. The critical assumptions are (a) constant construction costs per mile, (b) constant travel cost per mile per traveller, and (c) use of shortest paths in the network for all travel.

These results are adequate to give a complete solution when $n=3$. For this case, the possible optimal network configurations are identified, each is shown actually to arise as the optimum for suitable combinations of problem data, and the computations necessary to choose among them are described. One of the results for general n is an upper bound (roughly $n^4/4$) on the number of nodes, other than the original n points, in an optimal network. Another is the determination of an explicit threshold, for the ratio of construction cost to travel cost, beyond which each "auxiliary node" will lie on exactly three links.

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OPTIMAL NETWORKS JOINING n POINTS IN THE PLANE

W.A. Horn

National Bureau of Standards*

1. INTRODUCTION

This report develops a number of results on the problem of connecting n points in the plane, with given travel demands between each pair, by a minimum-cost network. The critical assumptions are (a) constant construction costs per mile, (b) constant travel costs per mile per traveler, and (c) use of shortest paths in the network for all travel.

Stated mathematically, the problem is to construct a network, or graph, N, in the plane, connecting the original n points and such that the number

$$\gamma(N) = \lambda\ell(N) + \Sigma\{\lambda_{ij}\ell(P_{ij}):1 \leq i < j \leq n\}$$

is minimized, where λ is the cost per mile of construction (in some units), λ_{ij} is the cost per mile to travel from point i to point j, P_{ij} is a selected shortest path from i to j,

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* Supported by Northeast Corridor Transportation Project, Department of Transportation, USA. No official endorsement is implied.

and $\ell(\ )$ is the length function, i.e., ℓ(subgraph K) is the total length of subgraph K, defined as the sum of the lengths of all the arcs of K.

Obviously, the original set of points to be connected will be nodes, or vertices, of the network, and we shall call them initial nodes. Other nodes of the network will be called auxiliary nodes. In order to simplify the problem, we assume that every auxiliary node is of order ≥ 3, that is, that it has at least 3 arcs incident upon it, since otherwise it is clear that it could be eliminated from the network without loss. (If an auxiliary node were of order 2, it could be eliminated and its two arcs combined into 1, while if it were of order 1, it and its incident arc could both be eliminated.)

A network which has auxiliary nodes of order ≥ 3 and connects the set of initial nodes will be called feasible. A network which is feasible and minimizes $\gamma(N)$, above, will be called optimal. We shall call γ the value function throughout this paper and, for convenience, shall assume that

$$\lambda + \sum_{i,j} \lambda_{ij} = 1 .$$

(This corresponds to adjusting the unit of money.) We also require that $\lambda > 0$, whereas $\lambda_{ij} \geq 0$.

Initial nodes will often be designated p_i , and a selected shortest path between p_i and p_j may be denoted P_{ij} . Also, for convenience, we define a number

$$w_k = \lambda + \Sigma\{\lambda_{ij} : A_k \subset P_{ij}\}$$

as the <u>weight</u> carried by an arc A_k of the network. This corresponds to the cost of constructing and using the portion of the network A_k .

The problem considered is that of finding, or at least finding properties of, optimal networks joining any set of n points. Mathematically, this problem has its origins in the so-called "Weber Problem" ([3]), in which a point in the plane is to be found such that the sum of its distances to 3 fixed points in the plane is minimal. This has been generalized to the case of finding a point the sum of whose distances to n given points is minimal (see [2], for example), and to the so-called "Street Network" problem ([1]). The latter problem involves finding in the plane a tree of shortest length connecting a given set of n points; it has not been solved generally.

This paper presents a discussion of some characteristics of networks which are optimal for the problem under consideration, and a complete listing of all possible optimal networks for the case n = 3 , together with a method for finding the optimal solutions on 3 points. In the course of the discussion, a bound is derived for the number of auxiliary nodes in a network on n points, a result which can easily be extended to more general, non-planar networks.

2. OTHER NOTATION

Other conventions which will be used in this paper are the following. For any points A and B in the plane, AB will represent the line segment from A to B, $\overline{AB}$ the line containing A and B, and $\overrightarrow{AB}$ the ray with initial point A and passing through B. We also shall use $[A,B)$ and $(A,B]$ to denote the line segment from A to B except for point B or A, respectively (the half open segment). An angle ABC, with vertex B, will often be written as ABC rather than $< ABC$. Distance between points A and B will be denoted by $\|A-B\|$.

A <u>similarity transformation</u> is a transformation T of the (x,y)-plane into itself given by

$$T(x,y) = S(R(T_1(x,y))) ,$$

where

$$T_1 : x \to \pm x + a$$
$$y \to \pm y + b ,$$

$$R : x \to x \cos \theta + y \sin \theta$$
$$y \to x \sin \theta - y \cos \theta ,$$

and

$$S : x \to \alpha x$$
$$y \to \alpha y \quad (\alpha > 0) .$$

It is well known that similarity transformations preserve angles and multiply all lengths by a fixed quantity. The ratio of length of $T(AB)$ to length of AB will be called simply the "ratio" of T. (Note that this ratio is equal to the α of S, above.)

3. CHARACTERISTICS OF OPTIMAL NETWORKS

We now derive some results concerning optimal networks joining n points. In the next section we will consider the special case of $n = 3$.

Lemma 1. For any optimal network N,

$$\gamma(N) = \sum_i w_i \ell(A_i) ,$$

where the summation is taken over all arcs A_i of N.

Proof. We have

$$\begin{aligned} \gamma(N) &= \sum_{j,k} \lambda_{jk} \ell(P_{jk}) + \lambda \ell(N) \\ &= \sum_{j,k} \lambda_{jk} \Big[\sum_{A_i \subset P_{jk}} \ell(A_i) \Big] + \lambda \sum_i \ell(A_i) \\ &= \sum_i \Big(\sum_{A_i \subset P_{jk}} \lambda_{jk} + \lambda \Big) \ell(A_i) \\ &= \sum_i w_i \ell(A_i) . \end{aligned}$$

Lemma 2. In an optimal network, all arcs are straight line segments.

Proof. Consider any arc A_i of N, an optimal network. Since $\lambda > 0$ by assumption, $w_i > 0$. If A_i is not a straight line, then replacing A_i by a straight line joining its endpoints decreases $\ell(A_i)$. By lemma 1, $\gamma(N)$ is thereby decreased, a contradiction.

Lemma 3. Any optimal network is connected.

Proof. If N is an optimal network which is not connected, then all of the initial nodes of N must lie in some connected component N_1, since otherwise some pair of initial nodes would not be pathwise connected, contradicting the definition of an optimal feasible network. But then all other components of N may be deleted from N and, as in lemma 2, $\gamma(N)$ will be decreased.

Lemma 4. Every arc of an optimal network is part of some minimal path.

Proof. If some arc A_i lies on no minimal path, then removing A_i from the network reduces $\gamma(N)$ by $\lambda\ell(A_i)$.

Lemma 5. Every optimal network is contained in the convex hull of its initial nodes.

Proof. Let $C = co(\{p_j\})$, the convex hull of the set of initial nodes $\{p_j\}$. Define a mapping P of the plane onto C as follows. For $x \in C$, $P(x) = x$. For $x \notin C$, let $y \in C$ be a point such that $\|x-y\|$ is the distance from x to C. Such a point exists, since C is compact. Furthermore, y is uniquely determined, since if $y' \neq y$ is another point of C for which $\|x-y\| = \|x-y'\|$, then

$$\|x-\tfrac{1}{2}(y+y')\| \leq \tfrac{1}{2}(\|x-y\| + \|x-y'\|) = \|x-y\| ,$$

with strict inequality holding for all cases in which $x, y,$ and y' do not lie on the same straight line. (This will always be the case if $\|x-y\| = \|x-y'\|$.) But $\frac{1}{2}(y+y')$ lies in C, since C is convex, and hence

$$d(x,C) \leq \|x-\tfrac{1}{2}(y+y')\| < \|x-y\| ,$$

a contradiction.

Define $P(x) = y$ in this case.

Since C is an r-sided convex polygon, it is clear that $P(x)$ lies in some side, or is some vertex, of this polygon whenever x is exterior to C. If $P(x)$ lies in the interior of some side, then $x-P(x)$ is perpendicular to this side. Thus the set of points in the exterior of C which are mapped into some side S_K by P is just the set of all half lines emanating from and perpendicular to S_K. Call this set E_K. Also, let F_i denote the set of all points mapped into vertex p_i. (See figure 1.)

Now consider the network $P(N)$, where N is any optimal network for the p_i. Since P is obviously continuous, all paths connecting nodes of N are transformed by P into paths connecting the same nodes in $P(N)$. Thus $P(N)$ is a feasible network for

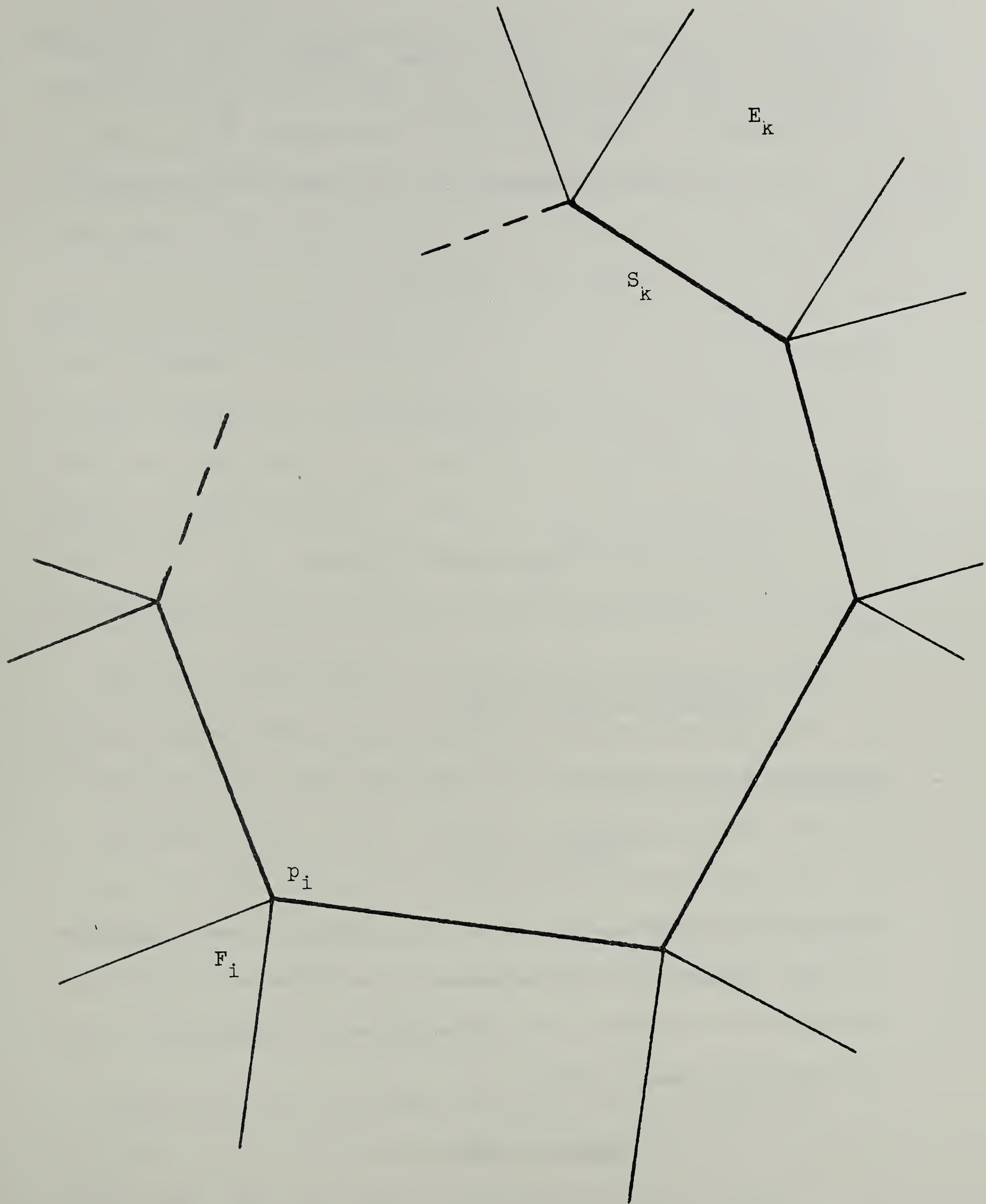

Figure 1. Illustration for Lemma 5.

the set of points $\{p_j\}$. We now show $\gamma(P(N)) < \gamma(N)$ if N contains points in the exterior of C .

Let L be any line segment of N in the exterior of C . Then

$$L = \bigcup_k (L \cap E_k) \cup \bigcup_i (L \cap F_i)$$

and so

$$\ell(L) = \sum_k \ell(L \cap E_k) + \sum_i \ell(L \cap F_i) .$$

But

$$P(L) = \bigcup_k P(L \cap E_k) \cup \bigcup_i \ell(L \cap F_i)$$

and

$$\ell(P(L)) = \sum_k \ell(P(L \cap E_k)) .$$

From this it is clear that

$$\ell(P(L)) \leq \ell(L)$$

with equality holding only if L lies in some E_k and is parallel to S_k . However, if all line segments in the exterior of C satisfied this condition, then N would not be connected, contradicting lemma 3. Thus

$$\gamma(P(N)) < \gamma(N) ,$$

contradicting the optimality of N . This completes the proof.

Theorem 6. If an optimal network N has n initial nodes, then each initial node is of order $\leq n-1$ and each auxiliary node is of order $\leq n$. Furthermore, every arc incident on an initial node must carry at least one minimal path from that node to some other initial node in any assignment of minimal paths.

Proof. Let p be any node of an optimal network N with some given assignment of minimal paths. Let $A_1, A_2, \ldots, A_m$ denote the arcs incident at p. Then by lemma 4 each A_i carries at least one minimal path P_{jk} going from p to (say) initial node p_j. Thus for each j from 1 to n, it is possible to define $f(j) = A_i$, where A_i is some arc at p carrying a minimal path to p_j, or $f(j) = \emptyset$ if no path from p_j contains p.

Now if more arcs leave p than there are remaining (other than p) initial nodes, then clearly there exists some A_i not of the form $f(j)$. But the portion of any minimal path containing A_i and going to p_r may be changed to lie along the path from p to p_r via $f(r)$, since it is clear that the latter path must be minimal between p and p_r. After all such changes of minimal path have been made for A_i, no minimal path will contain A_i. Thus, by lemma 4, N is not optimal.

From this, it is seen that the number of arcs incident at p is at most n if p is an auxiliary node and $n-1$ if p is an initial node. The last part of the theorem follows from the

statement that if $p = p_j$, an initial node, then the fact that some A_i incident at p_j carries no path P_{jk} implies that all paths entering p_j via A_i may be shifted to other arcs, as above, again contradicting lemma 4. This completes the proof.

Lemma 7. Let p be an auxiliary node in an optimal network with arcs (straight lines) $A_1, A_2, \ldots, A_m$ incident at p. Impose a coordinate system with origin at p. Let θ_i be the angle made by A_i with the positive x axis and w_i the weight carried by arc A_i.

Then

$$\sum_{i=1}^{m} w_i \cos \theta_i = 0 .$$

Proof. Consider the diagram of figure 2, where ℓ_i is the length of arc A_i, Δx a distance from p_1 to a new point p' along the x-axis, ℓ_i' the length of a new line A_i' drawn from the other vertex of A_i' to p', and $\Delta\theta_i$ the angle between A_i and A_i'. Then, by the Law of Sines,

$$\frac{\ell_i'}{\sin \theta_i} = \frac{\ell_i}{\sin(\theta_i + \Delta\theta_i)} = \frac{\Delta x}{\sin \Delta\theta_i} ,$$

or

$$\ell_i = \Delta x \, \frac{\sin(\theta_i + \Delta\theta_i)}{\sin \Delta\theta_i} ,$$

$$\ell_i' = \Delta x \, \frac{\sin \theta_i}{\sin \Delta\theta_i} .$$

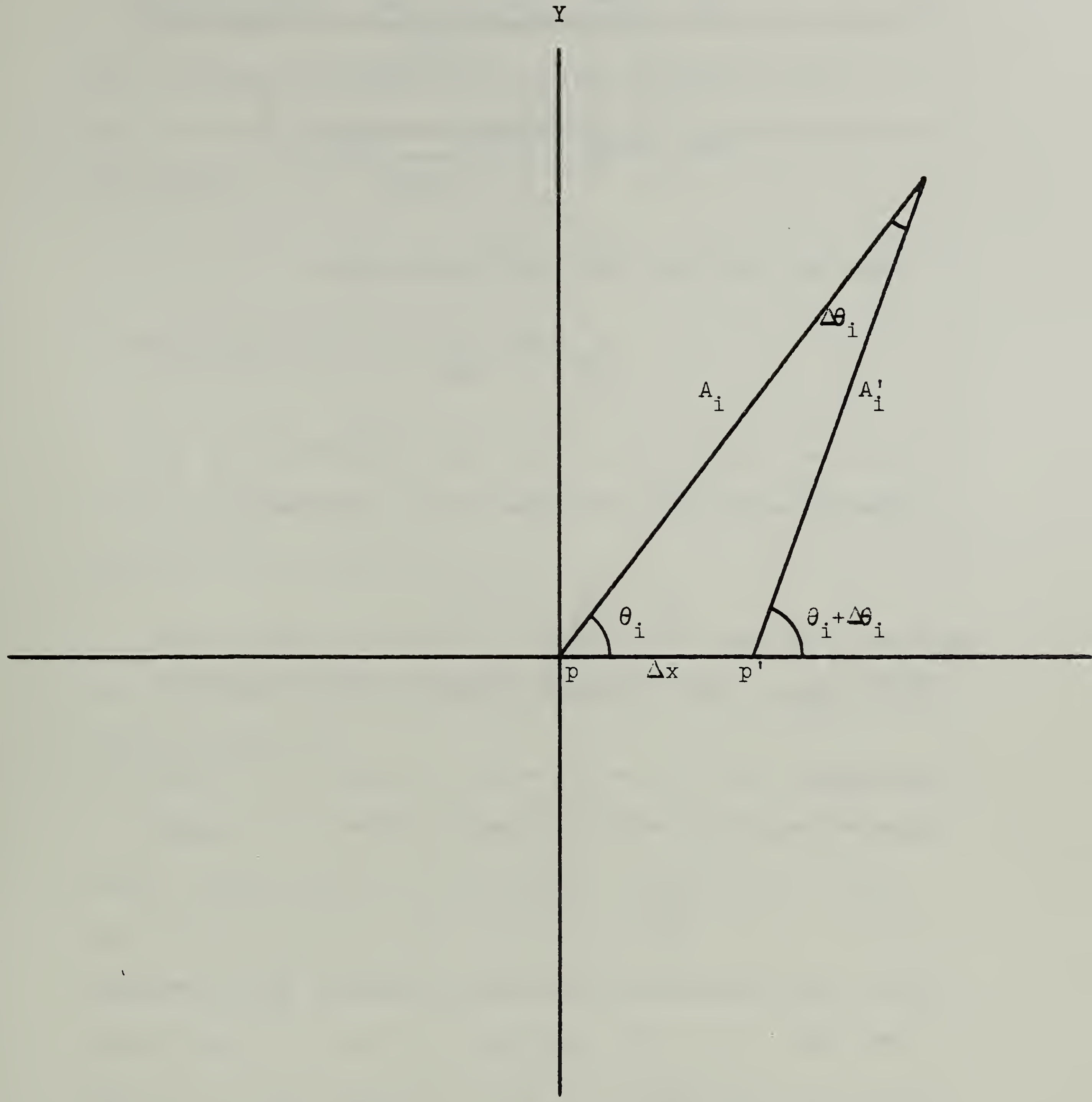

Figure 2. Illustration for Lemma 7.

Thus $\ell_i' - \ell_i = \Delta x \dfrac{\sin \theta_i - \sin \theta_i \cos \Delta\theta_i - \cos \theta_i \sin \Delta\theta_i}{\sin \Delta\theta_i}$

$$= \Delta x \left(-\cos\theta_i + \frac{\sin\theta_i (1-\cos\Delta\theta_i)}{\sin\Delta\theta_i} \right) .$$

Letting Δx, and hence $\Delta\theta_i$, go to 0, we get

$$\frac{d\ell_i}{dx} = -\cos\theta_i .$$

It is clear that the total directional derivative of γ with respect to moving p in the direction x will be

$$\frac{d\gamma}{dx} = \sum_{i=1}^{m} w_i(-\cos\theta_i) .$$

Setting $\frac{d\gamma}{dx} = 0$ for optimality gives the required equation.

Corollary 8. Let $\bar{v}_i$ be the vector of length w_i and angle θ_i, where w_i and θ_i, are as in lemma 7. Then

$$\sum_{i=1}^{m} \bar{v}_i = 0 .$$

Proof. By rotating the coordinate axes through $\frac{\pi}{2}$ in lemma 7, we have also

$$\sum_{i=1}^{m} w_i \sin\theta_i = 0 .$$

[Note that the results of lemma 7 and corollary 8 are not new. A slightly different proof of them is given in [2] , where it is also shown that the placement of p which minimizes the function

$$\sum_i w_i \|p-p_i\|$$

is uniquely given as that point where

$$\sum_i \bar{v}_i = 0 ,$$

if such a point exists. If no such point exists, the optimal placement of p is at a vertex.]

Next we turn our attention to more general networks containing arcs with given positive lengths assigned to them. We call such networks "measured".

<u>Lemma 9</u>. Let N be any measured network, not necessarily planar. Suppose that for some subset S of nodes it is true that the minimal path connecting any two nodes of S is unique. If any two nodes p and p' are contained in each of two minimal paths P_1 and P_2 connecting nodes of S , then the part of P_1 which connects p and p' is identical with the part of P_2 which connects p and p' .

Proof. If there were two paths between p and p', one a part of P_1 and the other a part of P_2, then each would have to be the same length, by the minimality of P_1 and P_2. But then either could be used as the part of P_1 or P_2 connecting p and p', contradicting the uniqueness of P_1 and P_2.

Theorem 10. Let N be any measured network, not necessarily planar. For any subset of nodes S, it is possible to choose minimal paths connecting pairs of nodes in S so that if any two nodes p and p' of N lie on minimal paths P_1 and P_2, then the part of P_1 which connects p and p' is identical with the part of P_2 which connects p and p'.

Proof. Let η be the minimum (positive) difference in length between a shortest path and a next shortest path connecting any pair of points in S, and let m be the total number of arcs of N. Define

$$\epsilon_i = \eta \cdot 2^{-i} , \quad i = 1,2,\ldots,m,$$

and let

$$\ell_i' = \ell_i + \epsilon_i ,$$

where ℓ_i is the length assigned to arc A_i. If we assign the new length ℓ_i' to arc A_i, it is clear that no two paths joining any pair of points in S can both be minimal and that any path

joining two points of S which is minimal under the new length assignment was also minimal under the old assignment. By lemma 9, the assignment of minimal paths using the new lengths satisfies the statement of the theorem.

We now use theorem 10 to obtain a gross upper bound on the number of nodes in any optimal planar network. This will then allow a proof of the existence of an optimum configuration for any set of initial nodes.

<u>Theorem 11</u>. If N is an optimal planar network on n initial nodes, then the number of auxiliary nodes of N is bounded by

$$n_a \leq \tfrac{1}{4}n(n-1)^2(n-2) \ .$$

<u>Proof</u>. Choose minimal paths in N satisfying the conditions of theorem 10. If P_{ij} and $P_{k\ell}$ are any two minimal paths meeting at some auxiliary node p of N, then it is clear that P_{ij} and $P_{k\ell}$ must exhibit exactly one of these behaviors:

(a) "Continue" at p. In this case two arcs incident at p both carry P_{ij} and $P_{k\ell}$.

or

(b) "Separate" at p. In this case, one arc incident at p carries P_{ij} and $P_{k\ell}$, another carries only P_{ij}, and a third carries only $P_{k\ell}$.

or

(c) "Cross" at p . In this case two separate arcs incident at p carry P_{ij} and two other arcs carry $P_{k\ell}$.

But from theorem 10 it is clear that $P_{ij} \cap P_{k\ell}$ is connected, and hence there is at most one point at which P_{ij} and $P_{k\ell}$ can cross and at most two points at which they can separate. In fact, if $\{i,j\} \cap \{k,\ell\} \neq \emptyset$, then there is at most one point at which they can separate (they cannot cross at an auxiliary node), since the two paths must be common at their common initial node.

We now count auxiliary nodes on a given minimal path P_{ij} at which P_{ij} and some other path $P_{k\ell}$ either cross or separate. If $\{i,j\} \cap \{k,\ell\} = \emptyset$, there are at most two such points and at most $\frac{1}{2}(n-2)(n-3)$ such $\{k,\ell\}$ sets. Thus there are at most $(n-2)(n-3)$ such auxiliary nodes. If $\{i,j\} \cap \{k,\ell\} \neq 0$, there is at most one such point, with $2(n-2)$ possible $\{k,\ell\}$ sets. Thus, altogether there are at most

$$(n-2)(n-3) + 2(n-2) = (n-1)(n-2)$$

auxiliary nodes on P_{ij} at which P_{ij} and some other path cross or separate.

Now every auxiliary node must be at a place where some two paths cross or separate, for every arc carries some minimal path,

by lemma 4, and if no minimal paths crossed or separated at an auxiliary node p then there would be only two arcs incident at p, a possibility ruled out by our original definition of the network. Thus if we count the total number of possible nodes of crossing or separation on all minimal paths, we will have at least twice the total number of auxiliary nodes n_a since each node is counted on at least 2 paths. Since there are $\frac{1}{2}(n)(n-1)$ paths P_{ij}, we have by the above remarks

$$n_a \leq \tfrac{1}{2}(\tfrac{1}{2})(n)(n-1)(n-1)(n-2)$$

$$= \tfrac{1}{4}n(n-1)^2(n-2) \; .$$

This completes the proof.

It should be noted in passing that this is not quite the best bound for n_a, even for general networks satisfying only the requirements of theorem 10. However, it is sufficient for our purposes here[1].

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(1) It has been shown that if N is a (possibly non-planar) network with paths between pairs of initial nodes satisfying the conditions given in the conclusion of theorem 10, then

$$n_a \leq \tfrac{1}{4}n(n-2)(n^2-4n+7) \; .$$

Theorem 12. For any given set of initial nodes there exists an optimal network.

Proof. For any given number of auxiliary nodes n_a and any configuration of arcs and assignment of minimal paths, we have

$$\gamma(N) = \Sigma\{w_i\|p_j-p_k\|:p_j,p_k \epsilon A_i \ , \ i = 1,2,\ldots,m\} \ ,$$

where the sum is taken over all arcs A_i of the network, p_j and p_k being the endpoints of A_i. Clearly $\gamma(N)$ is a continuous function, for this configuration and path assignment, of the auxiliary node positions, which in turn vary over the compact set C^{n_a}, the topological product of n_a replicates of the convex hull of the initial nodes. (See lemma 5.) Thus for each configuration and path assignment there is a minimum value for $\gamma(N)$. But for a fixed n_a, there is a finite number of ways in which auxiliary and initial nodes can be connected, and hence a finite number of configurations and path assignments. By theorem 11 there is a finite number of values for n_a. Hence there exists an optimal network.

The next section will deal with networks on 3 initial nodes and their various possible forms.

4. OPTIMAL NETWORKS FOR 3 POINTS

In this section we shall use the previous results to develop theorems about the form of optimal networks joining 3 points in the plane. Since it is clear that the optimal network joining collinear points is the straight line segment joining them, by lemma 5, we shall restrict our attention to non-collinear sets of 3 points.

Lemma 13. Suppose that in an optimal network N on n points some initial node p_i is of order 1. Let x be any point in the arc (line) incident at p_i. Then the network $N-(xp_i]$ is optimal for the set of initial nodes $\{p_j\}_{j \neq i} \cup \{x\}$ (where $(xp_i]$ is the half-open line segment from x to p_i), with demands the same as in the original, except that the demand from x to any other initial node p_j is λ_{ij}.

Proof. If N' is any network other than $N-(xp_i]$ which is optimal for $\{p_j\}_{j \neq i} \cup \{x\}$, then

$$\gamma(N') \geq \gamma(N-(xp_i]) ,$$

since, if $\gamma(N') < \gamma(N-(xp_i])$, then $\gamma(N' \cup (xp_i]) < \gamma(N)$, contradicting the optimality of N. Thus $N-(xp_i]$ is optimal for $\{p_j\}_{j \neq i} \cup \{x\}$.

Lemma 14. Suppose that N is an optimal network for the initial nodes $\{p_i\}$. If T is any similarity transformation, then $T(N)$ is optimal for the set of initial nodes $\{T(p_i)\}$ with demands λ_{ij} between $T(p_i)$ and $T(p_j)$.

Proof. Let $\gamma(N') < \gamma(T(N))$ for some N', with respect to the set $\{T(p_i)\}$. Then

$$\gamma(T^{-1}(N')) = \frac{1}{\rho}\gamma(N') < \frac{1}{\rho}\gamma(T(N)) = \gamma(N) ,$$

where ρ is the ratio of the transformation T. Thus $T^{-1}(N')$ forms a network connecting the points $\{p_i\}$ which gives a lower value than N for γ, a contradiction.

Theorem 15. If each initial node in an optimal network on 3 points is of order 2, then the network is a triangle connecting the 3 initial nodes.

Proof. Choose minimal paths in the network N to satisfy theorem 10. Consider any initial node of N, say p_1. By theorem 6, one of the two arcs incident at p_1 must carry path P_{12} and the other must carry P_{13}. By theorem 10, these paths can never meet after separating at p_1 and hence have an intersection of length 0. Similarly for the pairs of paths P_{12} and P_{23}, at p_2, and P_{13} and P_{23}, at p_3. Thus, since the intersection of any two minimal paths is of length 0, we must have

$$\gamma(N) \geq \sum_{i,j} (\lambda+\lambda_{ij})\ell(P_{ij}) .$$

But clearly $\sum_{i,j} (\lambda+\lambda_{ij})\ell(P_{ij})$ is minimized when each $\ell(P_{ij})$ is minimized, which is possible only when N is a triangle joining the 3 initial nodes. This completes the proof.

Theorem 16. The only topological configurations possible for an optimal network on 3 nodes are those shown in figure 3.

Proof. We consider networks on three points according to the orders of their initial nodes. It is clear that each initial node must be of order ≥ 1, and by theorem 6 the order must be ≤ 2.

First assume all nodes are of order 2. Then by the previous theorem, we have figure 3a, the triangle.

If two nodes are of order 2 and one of order 1, consider the new set of nodes formed by deleting the node of order 1, p_i, and adding the auxiliary node p at the other end of the arc incident at p_i. By lemma 13, the network $N-(pp_i]$ must be optimal for this new set of nodes (with the proper demands). But p is of order 2, since it was previously of order 3, by theorem 6, and now has 1 less arc. By the previous theorem, $N-(pp_i]$ is therefore a triangle, and so N is as in 3c.

(Note that P must be an auxiliary node in the above proof, since if $p = p_j$ then the resulting network $N-(pp_i]$ is a straight line segment, by lemma 5, thereby contradicting the fact that the other initial node is of order 2).

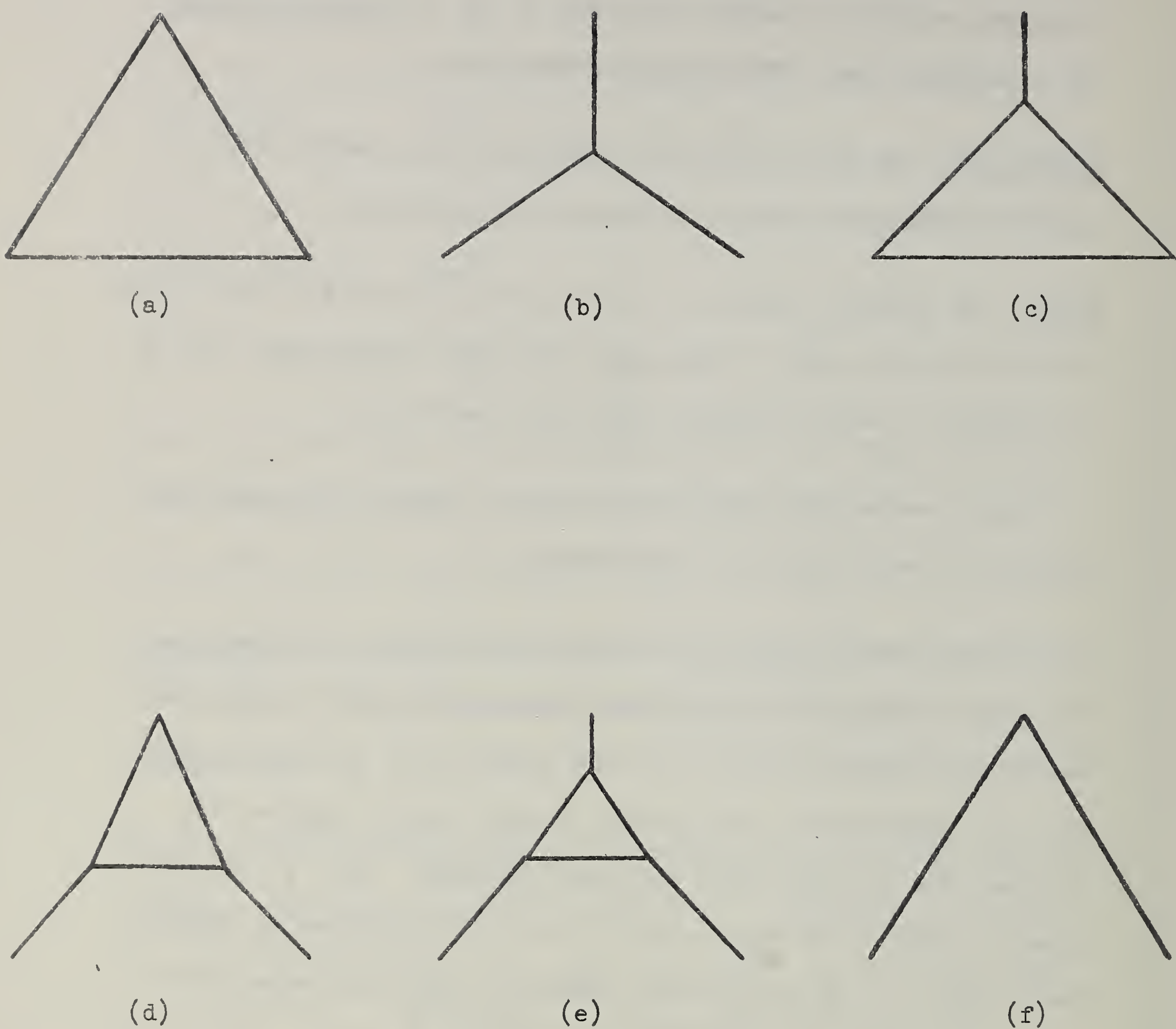

Figure 3. Illustration for Theorem 16.

Next consider the case where two initial nodes say p_1 and p_2 , are of order 1 and the third, p_3 , is of order 2. Let p' be the node at the other end of the arc incident at p_1 and p'' the node at the other end of the arc incident at p_2 . Form the network $N-(p'p_1] - (p''p_2]$. By a double application of lemma 13, this new network must be optimal for the set of points p' , p'' , and p_3 , with appropriate demands. Now it may happen that $p' = p_3$, in which case it is clear that figure 3f is obtained. Or it may happen that p' , p'' , and p_3 are all different, in which case, by an argument similar to that above, all are of order 2. Hence $N-(p'p_1] - (p''p_2]$ is a triangle, and we get 3d. It is not possible, however, that one of p' and p'' , say p' , is equal to p_3 , while $p'' \neq p_3$. For then p'' is of order 2 in the new network, by theorem 6, but is of order 1 by lemma 5. Similarly, $p' \neq p''$.

Finally, consider the case where each initial node is of order 1. If p' , p'' , and p''' are the nodes at the other end of the arcs incident at p_1 , p_2 , and p_3 , respectively, then either $p' = p'' = p'''$ or p , p'' , and p''' are all different, by an application of lemma 5 and theorem 6 similar to the above. Thus we get either 3b or 3e . This completes the proof.

The topological configurations for networks given in 3a, 3b, 3c, 3d, 3e, and 3f will be called suggestively the Δ, Y, $\bar{Y}$, A, Δ^2, and V configurations. It will be the work of the next two proofs to show that the Δ^2 is an inessential configuration, that is, that it need not be considered when looking for optimal networks.

<u>Lemma 17</u>. Given any triangle ABC, with lines $\overline{BD}$ and $\overline{CE}$ passing through vertices B and C and the sides opposite them, respectively (see figure 4), for any point A' interior to the triangle and close enough to A, there exist points $B' \in BD$ and $C' \in CE$ such that triangle A'B'C' is similar to ABC, with vertices in the order given. Furthermore, if $\{A^{(n)}\}$ is a set of points interior to ABC, with $A^{(n)} \to A$, and $\{A^{(n)}B^{(n)}C^{(n)}\}$ is a set of triangles satisfying the above statements, then $B^{(n)} \to B$ and $C^{(n)} \to C$.

<u>Proof</u>. Let A' be any point interior to quadrilateral ADFE (figure 5). For each point $B' \in BD$ define a point $f(B')$ as follows. Draw line A'B' and construct at A" angle $B'A'G = BAC$. Let $f(B')$ be the intersection of $\overrightarrow{A'G}$ and $\overline{CE}$. (If $\overrightarrow{A'G}$ and $\overline{CE}$ do not intersect, then $f(B')$ is not defined.)

Now it is clear that $f(B)$ exists and is contained in CE, since when $B' = B$ line A'G makes a positive angle with respect to the horizontal (line $\overline{AC}$). In the same way, it is clear that

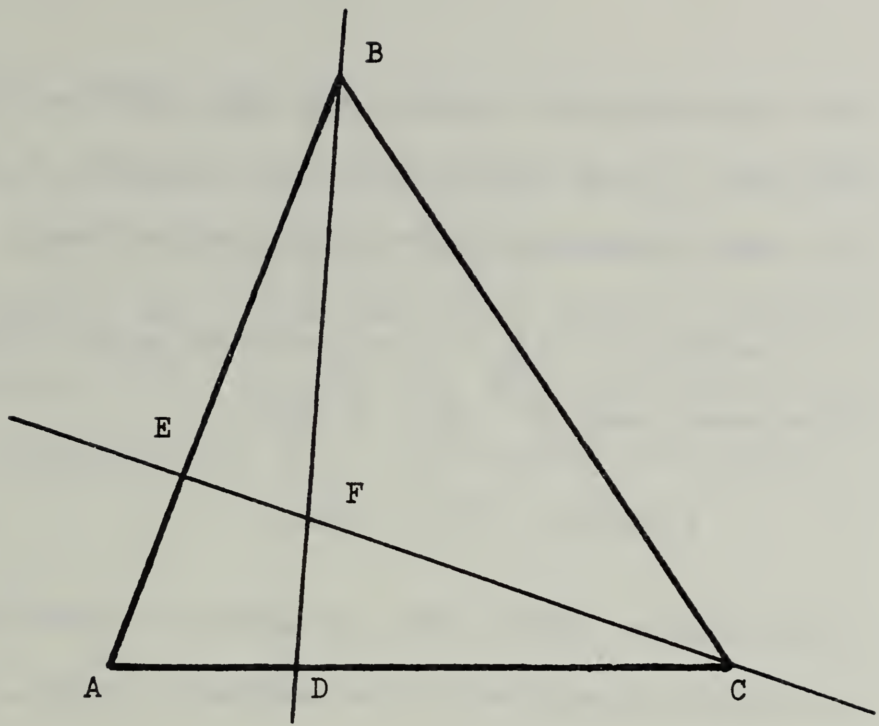

Figure 4. Illustration for Lemma 17.

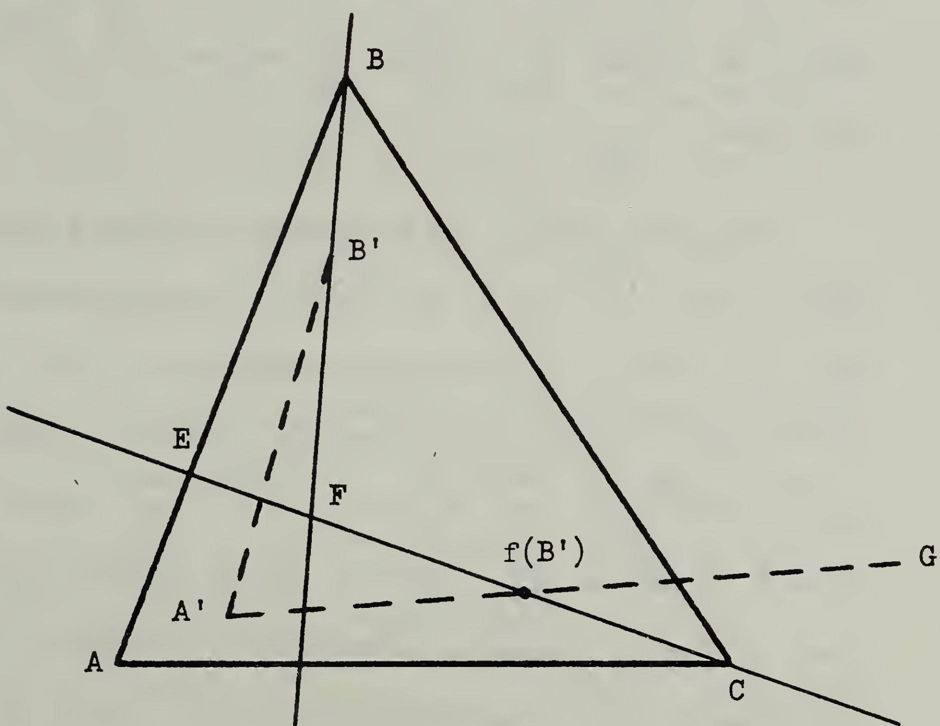

Figure 5. Illustration for Lemma 17.

there is a point B'' in BD such that $f(B'') = C$, since we may draw line $A'C$ and then construct an angle at A' equal to ABC , in a symmetric manner to the above construction.

Now when $B' = B$, it is clear that angle $A'Bf(B) < ABC$. Furthermore, when $B' = B''$, we have

$$B''f(B'')A' = B''CA' < BCA ,$$

so that, since $B''A'C = BAC$, we must have $A'B''C > ABC$. Now it is easily shown that f is continuous in B' . Therefore, there exists some B' lying between B and B'' such that

$$A'B'f(B') = ABC .$$

But this implies that triangle $A'B'f(B')$ is similar to ABC , and so we choose $C' = f(B')$ to demonstrate the first part of the lemma.

Next let $\{A^{(n)}\}$ be a sequence of points in the interior of $ADFE$ with $A^{(n)} \to A$. If $\{B^{(n)}\}$ is a corresponding sequence on BD and $\{C^{(n)}\}$ a corresponding sequence on CE , so that $A^{(n)}B^{(n)}C^{(n)}$ is always a triangle similar to ABC , as above, and if $B^{(n)} \not\to B$, the sequence $\{B^{(n)}\}$ must have a limit point $B_o \neq B$ lying in BD. In fact, we may assume, for convenience, that $B^{(n)} \to B_o$, since a subsequence converges to B_o in any event. Now clearly $C^{(n)}$ is continuous in $A^{(n)}$ and $B^{(n)}$. But if

line segment B_oA is constructed and an angle constructed at A equal to BAC, it is apparent that the triangle B_oAC_o containing B_oA as a side, B_oAC_o as an angle, and similar to BAC, has its vertex C_o outside BAC. Thus some $C^{(n)}$ must lie outside BAC, by continuity of $C^{(n)}$ in $A^{(n)}$ and $B^{(n)}$, and this contradicts the construction of $C^{(n)}$. This proves the second part of the lemma.

Theorem 18. Suppose that a Δ^2 network is optimal for some 3 points. Then there is a Y network which is optimal for these 3 points.

Proof. Let the Δ^2 network N be as shown in figure 6. (Note that lemma 5 implies that the smaller triangle must lie inside triangle ABC.) By the previous lemma, there exist points A', B', and C' on lines $\overline{AA''}$, $\overline{BB''}$, and $\overline{CC''}$, respectively, such that triangle $A'B'C'$ is similar to triangle ABC. By the second part of lemma 17, these points may be chosen so that $A' \epsilon AA''$, $B' \epsilon BB''$, and $C' \epsilon CC''$.

Let T be a similarity transformation of the plane such that $T(A) = A'$, $T(B) = B'$, and $T(C) = C'$. By lemma 14, $T(N)$ is an optimal network for the points A', B', and C', with the same demands as those of A, B, and C respectively.

But so is $N-[AA') - [BB') - [CC')$, by repeated applications of lemma 13. Hence

$$\gamma(T(N)) = \gamma(N-[AA')-[BB')-[CC')).$$

Thus if

$$N' = T(N) \cup AA' \cup BB' \cup CC' ,$$

we have

$$\begin{aligned}\gamma(N') &= \gamma(T(N)) + \gamma(AA' \cup BB' \cup CC') \\ &= \gamma(N-[AA')-[BB')-[CC')) \\ &\qquad + \gamma(AA' \cup BB' \cup CC') \\ &= \gamma(N) .\end{aligned}$$

(See figure 7.) From this it follows that N' is also an optimal network for A,B, and C. But

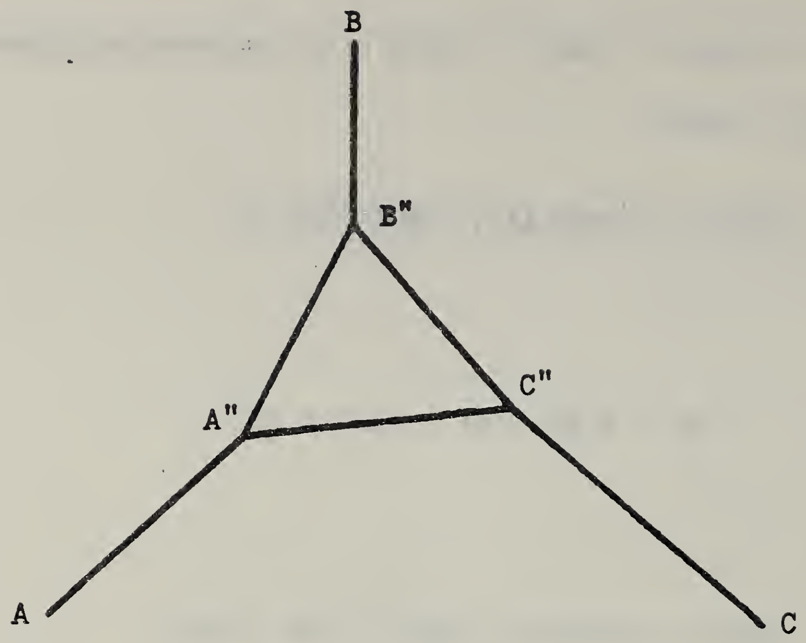

Figure 6. Illustration for Theorem 18.

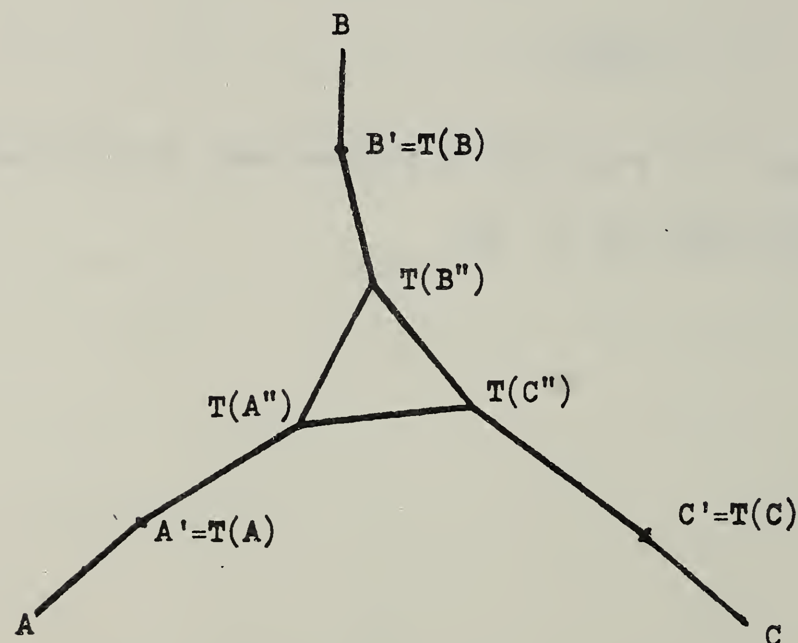

Figure 7. Illustration for Theorem 18.

this implies, by lemma 2, that $AA'T(A'')$, $BB'T(B'')$, and $CC'T(C'')$ are straight angles. Thus we have a new Δ^2 network N' where the diameter of the inner triangle has been multiplied by $\rho < 1$, the ratio of T.

It is clear that by following the same procedure, N' can be transformed to a network N'' in which the diameter of the inner triangle has been multiplied by ρ^2, and, in fact, this process can be continued so that at the m-th step we have a Δ^2 network whose inner triangle is of a diameter ρ^m times that of the original. But this implies that line segments AA'', BB'', and CC'', when extended, meet at some common point O, inside triangle ABC. Since $\gamma(N^{(m)}) = \gamma(N)$ for each stage, this implies that the Y network formed by lines AO, BO, and CO is also optimal. This completes the proof.

From theorem 18 it is seen that Δ^2 networks need not be considered when looking for an optimal network joining 3 points. This does not mean, of course, that there are no optimal Δ^2 networks.

The next two lemmas, although applying to the 3-point problem, are given in a more general setting because of their usefulness when trying to solve larger networks.

Lemma 19. Let any 3 nodes A, B, and C of an optimal network be joined by a Y configuration. If a, b, and c are the respective weights of the arcs of the Y incident at A, B, and C and α, β, and ζ are the respective angles of the Y opposite these arcs, then

$$\cos \alpha = - \frac{b^2+c^2-a^2}{2bc} ,$$

$$\cos \beta = - \frac{a^2+c^2-b^2}{2ac} ,$$

and

$$\cos \zeta = - \frac{a^2+b^2-c^2}{2ab} .$$

Proof. From corollary 8 we have that the weights a, b, and c form a triangle. Further, it is obvious that the construction of the triangle is such that the angle between any two adjacent sides is the supplement of the angle made by the arcs of the Y corresponding to those two sides. Suppose that α', β', and ζ' are the angles opposite sides of length a, b, and c, respectively, in the triangle. By the law of cosines,

$$\cos \alpha' = \frac{b^2+c^2-a^2}{2bc} ,$$

$$\cos \beta' = \frac{a^2+c^2-b^2}{2ac} ,$$

and

$$\cos \zeta' = \frac{a^2+b^2-c^2}{2ab} .$$

Since $\alpha' = \pi - \alpha$, $\beta' = \pi - \beta$, and $\zeta' = \pi - \zeta$, this completes the proof.

Lemma 20. Let any 3 nodes A , B , and C of an optimal network be joined by a V configuration, where one leg of the V is AB , the other BC . Let

$$S_A = \{\{i,j\} : \text{optimal path } P_{ij} \text{ contains } AB\} ,$$

$$S_C = \{\{i,j\} : \text{optimal path } P_{ij} \text{ contains } BC\} ,$$

$$a = \lambda + \Sigma\{\lambda_{ij}:\{i,j\} \in S_A\} ,$$

$$b = \lambda + \Sigma\{\lambda_{ij}:\{i,j\} \in (S_A-S_C) \cup (S_C-S_A)\} ,$$

$$c = \lambda + \Sigma\{\lambda_{ij}:\{i,j\} \in S_C\} .$$

Then

$$\cos ABC \leq - \frac{a^2+c^2-b^2}{2ac} .$$

Proof. Assume $\cos ABC > - \frac{a^2+c^2-b^2}{2ac}$. It is clear from the definition of a , b , and c that $a+b > c$, $a+c > b$, and $b+c > a$, and therefore there exists a triangle with sides of length a , b , and c . Let

$$\alpha = \cos^{-1}\left(-\frac{b^2+c^2-a^2}{2bc}\right) ,$$

$$\beta = \cos^{-1}\left(-\frac{a^2+c^2-b^2}{2ac}\right) ,$$

and

$$\zeta = \cos^{-1}\left(-\frac{a^2+b^2-c^2}{2ab}\right) ,$$

where $0 < \alpha , \beta , \zeta , < \pi$. Then clearly $\beta > ABC$. Thus $\alpha + \zeta + ABC < 2\pi$ and so we can find $\bar{\alpha}$ and $\bar{\zeta}$ such that $\bar{\alpha} + \bar{\zeta} = ABC$, $\alpha + \bar{\alpha} < \pi$, and $\zeta + \bar{\zeta} < \pi$.

Referring to figure 8, we construct line BO such that $ABO = \bar{\zeta}$, $CBO = \bar{\alpha}$. At O we construct $BOA' = \zeta$ and $BOC' = \alpha$. It is clear that $\overline{A'O}$ must intersect AB , since $\zeta + \bar{\zeta} < \pi$, and similarly $\overline{C'O}$ must intersect $\overline{BC}$. Also, it is obvious that, by choosing O close enough to B we can make A' and C' lie in segments AB and BC , respectively. Thus the figure is as shown.

Now consider the two networks connecting points A' , B , and C': the Y network A'O , BO, and C'O , and the V network A'B and BC' . We shall show that the Y gives a smaller value for γ .

It is clear that in the Y network joining A' , B , and C' the weights carried by arcs A'O , BO , and C'O are respectively, a , b , and c , since all optimal paths going from A' to C' (or A to C) do not use arc BO . From the proof of lemma 7, we

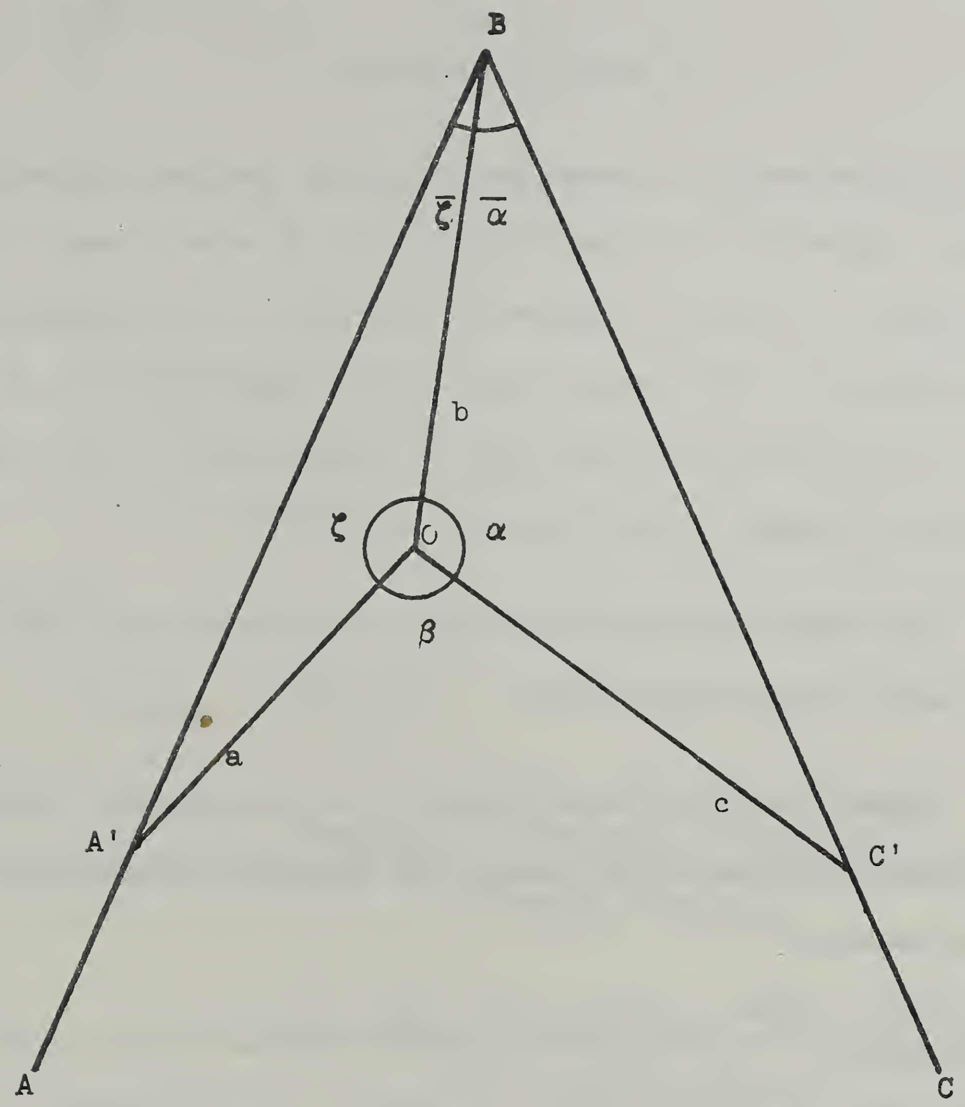

Figure 8. Illustration for Lemma 20.

see that the directional derivative of γ in the direction $\overrightarrow{OB}$ is given by

$$-b - a \cos \zeta - c \cos \alpha .$$

But this derivative is zero when O is in the position determined above. Therefore it becomes positive as O moves toward B, since both ζ and α increase. Therefore, γ increases as O approaches B. This shows that the V connecting A', B, C' is not optimal, and hence the V connecting A, B, and C cannot be optimal. This completes the proof.

Note that the remarks following corollary 8 could also have been used to prove this lemma.

Lemma 20 has an interesting corollary for general optimal networks, which restricts greatly the possible configurations for these networks.

<u>Corollary 21</u>. If $\lambda > \sqrt{2}-1$, then any optimal network contains no nodes of order > 3 and no triangles or quadrilaterals. Therefore every auxiliary node is of order 3.

<u>Proof</u>. Using the notation of the previous lemma, let a V join nodes A, B, and C, with B its vertex, and let

$$\mu_1 = \Sigma \{\lambda_{ij}: \{i,j\} \in S_A - S_C\} ,$$

$$\mu_2 = \Sigma \{\lambda_{ij}: \{i,j\} \in S_C - S_A\} ,$$

and

$$\mu_3 = \Sigma \{\lambda_{ij}: \{i,j\} \in S_A \cap S_C\} .$$

Then

$$a = \lambda + \mu_1 + \mu_3 ,$$
$$b = \lambda + \mu_1 + \mu_2 ,$$
$$c = \lambda + \mu_2 + \mu_3 .$$

By lemma 20,

$$\cos ABC \leq - \frac{a^2+c^2-b^2}{2\,ab}$$

$$= - \frac{(\lambda+\mu_1+\mu_3)^2 + (\lambda+\mu_2+\mu_3)^2 - (\lambda+\mu_1+\mu_2)^2}{2(\lambda+\mu_1+\mu_3)(\lambda+\mu_2+\mu_3)}$$

$$= - \frac{\lambda^2 + 2\mu_3^2 + 4\lambda\mu_3 + 2\mu_1\mu_3 + 2\mu_2\mu_3 - 2\mu_1\mu_2}{2(\lambda+\mu_1+\mu_3)(\lambda+\mu_2+\mu_3)}$$

$$\leq - \frac{\lambda^2 - 2\mu_1\mu_2}{2(\lambda+\mu_1+\mu_3)(\lambda+\mu_2+\mu_3)} .$$

But $\mu_1 + \mu_2 \leq 1 - \lambda$. Therefore,

$$2\mu_1\mu_2 \leq 2[\tfrac{1}{2}(1-\lambda)]^2 = \tfrac{1}{2}(1-\lambda)^2 ,$$

so that

$$\lambda^2 - 2\mu_1\mu_2 \geq \lambda^2 - \tfrac{1}{2}\, 1-\lambda)^2$$

$$> (\sqrt{2}-1)^2 - \tfrac{1}{2}(2-\sqrt{2})^2 = 0 .$$

Therefore, $\cos ABC < 0$, and $ABC > \frac{\pi}{2}$.

This implies that every V , or meeting of two arcs, has an angle $> \frac{\pi}{2}$ in an optimal network. Since every angle of a triangle or quadrilateral defines a V , and since every node defines V's between each pair of incident arcs, there can be no triangles or quadrilaterals and no nodes of order > 3 . This completes the proof.

Next we compare the V and the Δ .

<u>Lemma 22</u>. Let any 3 nodes A , B , and C of an optimal network be joined by a V with vertex B . If μ_3 is as in the proof of corollary 21 and $a' = \|B-C\|$, $b' = \|A-C\|$, and $c' = \|A-B\|$, then

$$\lambda \geq \mu_3 \left(\frac{a'+c'-b'}{b'}\right) .$$

<u>Proof</u>. Let μ_1 and μ_2 also be as defined in corollary 21. If N is the original network and N' is the network obtained by adding line AC to N , then since all minimal paths from B to C can now travel directly over AC in N' we have

$$0 \geq \gamma(N) - \gamma(N') \geq [c'(\lambda+\mu_1+\mu_3) + a'(\lambda+\mu_2+\mu_3)]$$

$$- [c'(\lambda+\mu_1) + a'(\lambda+\mu_2) + b'(\lambda+\mu_3)]$$

$$= (a'+c'-b')\mu_3 - b'\lambda .$$

Thus

$$\lambda \geq \mu_3 \left(\frac{a'+c'-b'}{b'}\right) .$$

So far nothing has been said about whether each of the possible configurations given in theorem 16 can actually occur in an optimal network for some set of initial nodes and some set of λ and λ_{ij}. The last lemma allows us to construct an A network which is actually optimal for a set of 3 points and, in fact, is the only type of optimal network for these points. This implies, by lemma 13, that there exist optimal Y's , $\bar{Y}$'s , Δ's , and V's (where certain demands between pairs are 0 for the Y and V).

Example 23. There exists an optimal A .

Demonstration. Let initial nodes A , B , and C be the vertices of a triangle with $\|A-B\| = \|A-C\| = 1$, angles $BAC = \frac{2\pi}{3}$ and $ABC = ACB = \frac{\pi}{6}$. (See figure 9.) Let $\lambda = 0.04$ and $\lambda_{AB} = \lambda_{BC} = \lambda_{AC} = 0.32$. We now show that the optimal network for A , B , and C is an A network.

First consider the V and the Y as possible networks. Since $\lambda + \lambda_{AB} + \lambda_{AC} = \lambda + \lambda_{AB} + \lambda_{BC} = \lambda + \lambda_{AC} + \lambda_{BC}$, the angles of a Y

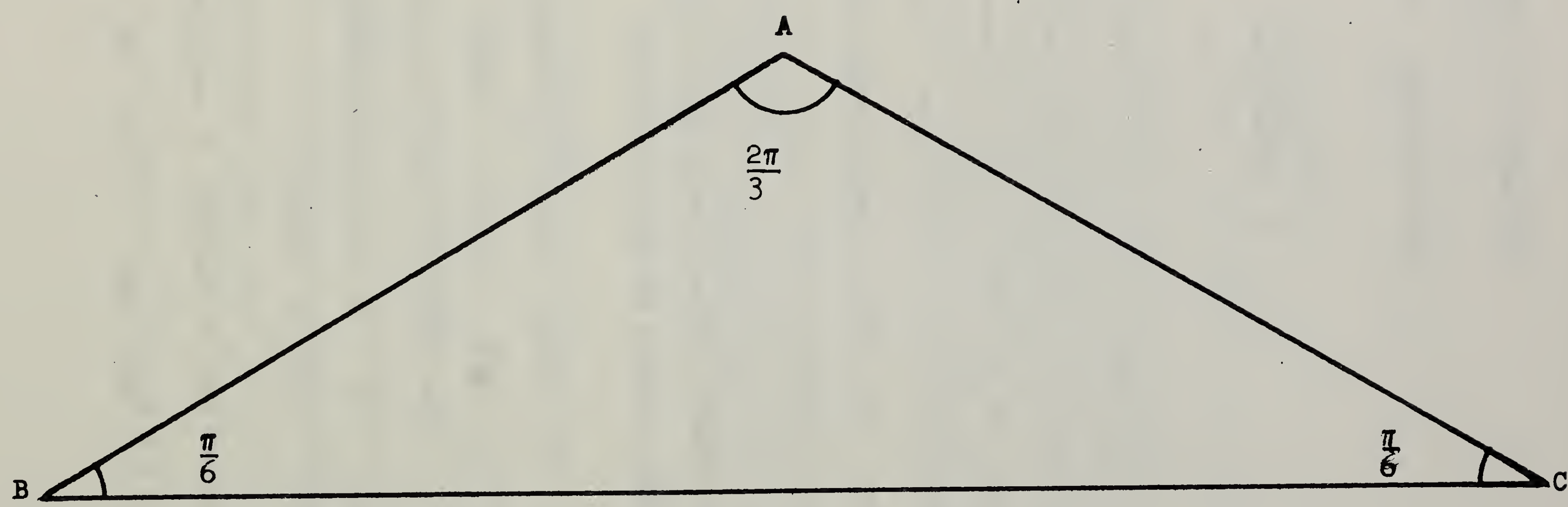

Figure 9. Illustration for Example 23.

must all be $\frac{2\pi}{3}$, or the angle of a V must be $\geq \frac{2\pi}{3}$, by lemmas 19 and 20. Thus a Y is impossible, since angle BAC $= \frac{2\pi}{3}$, and so we consider only the V which, by lemma 20, must have its vertex at A . But by lemma 22, if a V is optimal then

$$\lambda \geq \lambda_{BC} \; \frac{b'+c'-a'}{a'} \quad ,$$

where $a' = \|B-C\|$, $b' = \|A-C\|$, and $c' = \|A-B\|$. Since $b' = c' = 1$, we have $a' = \sqrt{3}$, and so

$$.04 \geq .32 \left(\frac{1+1-\sqrt{3}}{\sqrt{3}}\right) = .32 \left(\frac{2\sqrt{3}}{3} - 1\right) .$$

That is,

$$\frac{1}{8} \geq \frac{2\sqrt{3}}{3} - 1 .$$

But $\frac{1}{8} = .125$ and $\frac{2\sqrt{3}}{3} - 1 \cong .155$. Thus a V is not possible either.

Next consider a Δ or a $\overline{Y}$. Since a Δ has three initial nodes of order 2 and a $\overline{Y}$ has two such nodes, one of B and C must be a node of order 2 in a Δ or $\overline{Y}$. Thus if B, for instance, is such a node, and β is the angle made by the arcs incident at B , by lemma 20 we must have

$$\cos \beta \leq - \frac{(\lambda+\lambda_{AB})^2 + (\lambda+\lambda_{BC})^2 - (\lambda+\lambda_{AB}+\lambda_{BC})^2}{2(\lambda+\lambda_{AB})(\lambda+\lambda_{BC})}$$

$$= - \frac{(.36)^2 + (.36)^2 - (.68)^2}{2(.36)^2} \cong .78 .$$

But $\beta \leq \frac{\pi}{6}$, since $ABC = \frac{\pi}{6}$. Thus $\cos \beta \geq \frac{\sqrt{3}}{2} \cong .87$, a contradiction.

By theorems 16 and 18, the only other possible configuration for an optimal network is the A . This finishes the demonstration.

Finally, let us consider a method for actually finding an optimal network joining any given 3 points. If an electronic computer is available, this is a very quick procedure, but even if this is not the case a hand calculation can be done in a reasonable time. The method is to check the actual value of each of the five possible optimal configurations given in theorem 16 (excluding the Δ^2). For the V and Δ this is straightforward. For the Y and $\bar{Y}$, the method of locating the auxiliary node given in [2] is very quick, according to the authors. (Of course, for the $\bar{Y}$ the procedure is to find a Y connecting the points, but with demand 0 between the two points which are linked directly.) The A network is a bit more difficult to find, but a procedure may be used in which the position of each auxiliary node is alternately optimized, holding the other auxiliary node fixed and using the method of [2] . This is done until no further appreciable movement of either auxiliary node occurs.

As an example of how the angles of an A network might be computed, we solve for the values of example 23. (Admittedly, this example is simplified by its symmetry.)

Looking at figure 10, where the A network is outlined in black with numbers along its various arcs representing their weights, we see by lemma 19 that angle α must satisfy

$$\cos \alpha = - \frac{(.36)^2+(.36)^2-(.68)^2}{2(.36)^2} = .784 .$$

Thus $\alpha = 38.4^\circ$

From this, $\beta = 180 - 2\alpha = 103.2^\circ$. From symmetry, it seems likely that the two angles represented by γ should be equal. Thus

$$\gamma = \tfrac{1}{2}(120-\beta) = 8.4^\circ .$$

The angles represented by δ's will then be equal, and we have

$$\delta = \tfrac{1}{2}(360-\alpha) = 160.8^\circ ,$$

$$\epsilon = 180 - (\gamma+\delta) = 10.8^\circ ,$$

$$\zeta = 30 - \epsilon = 19.2^\circ .$$

That these are the actual angles can be seen by the fact that that if γ and ϵ are constructed at A and B and at C and B , then the triangles formed are congruent. Thus it is clear that the middle line of the A is parallel to BC , so that the angles α are equal in this construction. Since it can be shown that the positioning of auxiliary nodes is unique for optimality here, the angles are as given.

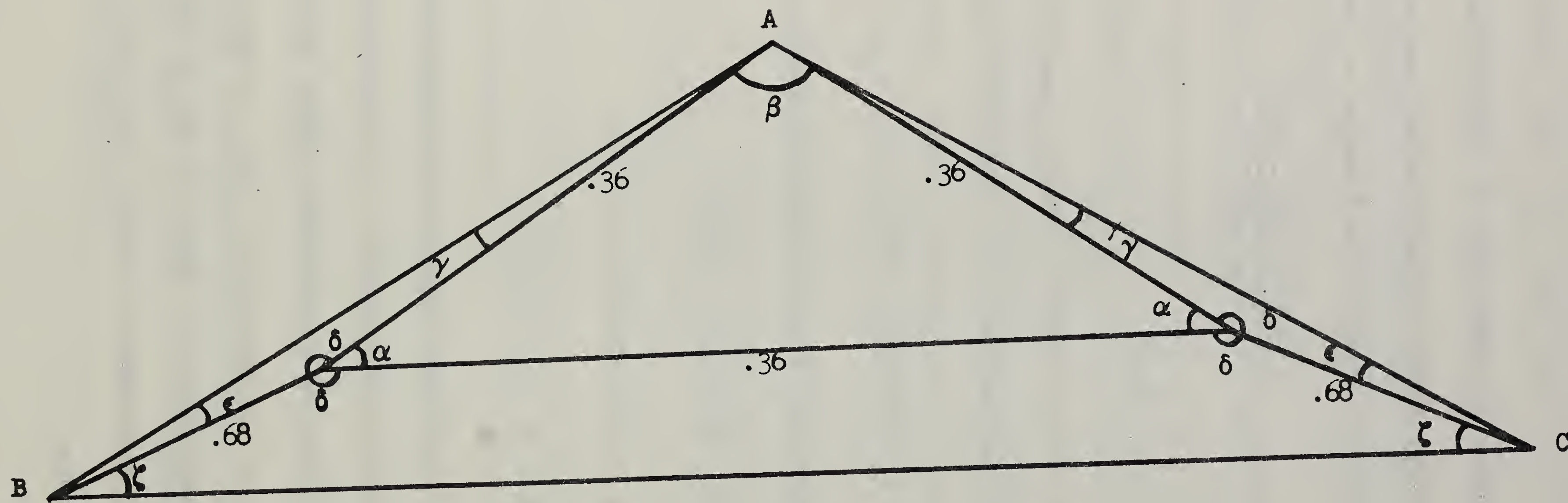

Figure 10. Illustration for Calculation of Optimal "A"

5. FURTHER WORK

It is clear that the surface has merely been scratched by the work here, and that much more remains to be done. Although optimal networks for 3 points can be found with relative ease, when $n = 4$ the problem is much more difficult, and for larger values of n there is no easy way to find the optimal network, even with a high-speed computer. Of course, if the configuration of the optimal network and optimal paths is known, then a generalized version of the procedure for finding an optimal A network, given in the last section, can be used. However, there are a great many different configurations which can exist.

One direction for more work which seems promising is to group networks according to the number of cycles they possess (the Betti number). The author has found (although it is not proved in this paper) that the networks on 3 points with one cycle (the Δ, $\overline{Y}$, and A) are <u>mutually exclusive</u> in the sense that if one of these types is optimal then there can be no optimal network of one of the other types. The same can be said for the V and Y. Thus there is some hope that a certain exclusiveness might exist among networks with the same Betti number. Of course, this may prove not to be of much help anyway, since no general way is known to solve the pure shortest-network problem ($\lambda=1$, all $\lambda_{ij}=0$) on n points, even though it is known that the network must be a tree. ([1])

Another approach is to consider the ways in which optimal paths split apart, join together, or cross. Theorem 10 states that optimal paths may always be chosen in such a way that, once two paths separate, they never again meet. This may give some insight into possible configurations for an optimal network.

6. REFERENCES

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